Determinant

Linear Algebra

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Overview

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Introduction

Determinant of a matrix

The determinant of a 2×2 matrix $A = [a_{ij}]$ is the number: Why???

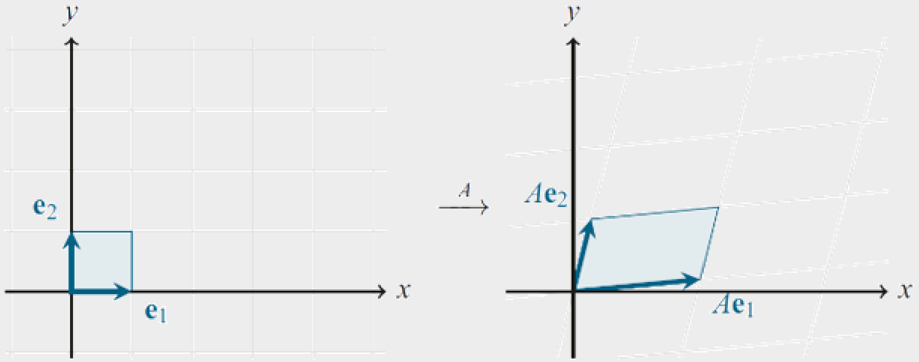
$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

- The absolute value of the determinant of a matrix measures how much it expands space when acting as a linear transformation. That is, it is the area (or volume, or hypervolume, depending on the dimension) of the output of the unit square, cube, or hypercube after it is acted upon by the matrix.

Geometric interpretation

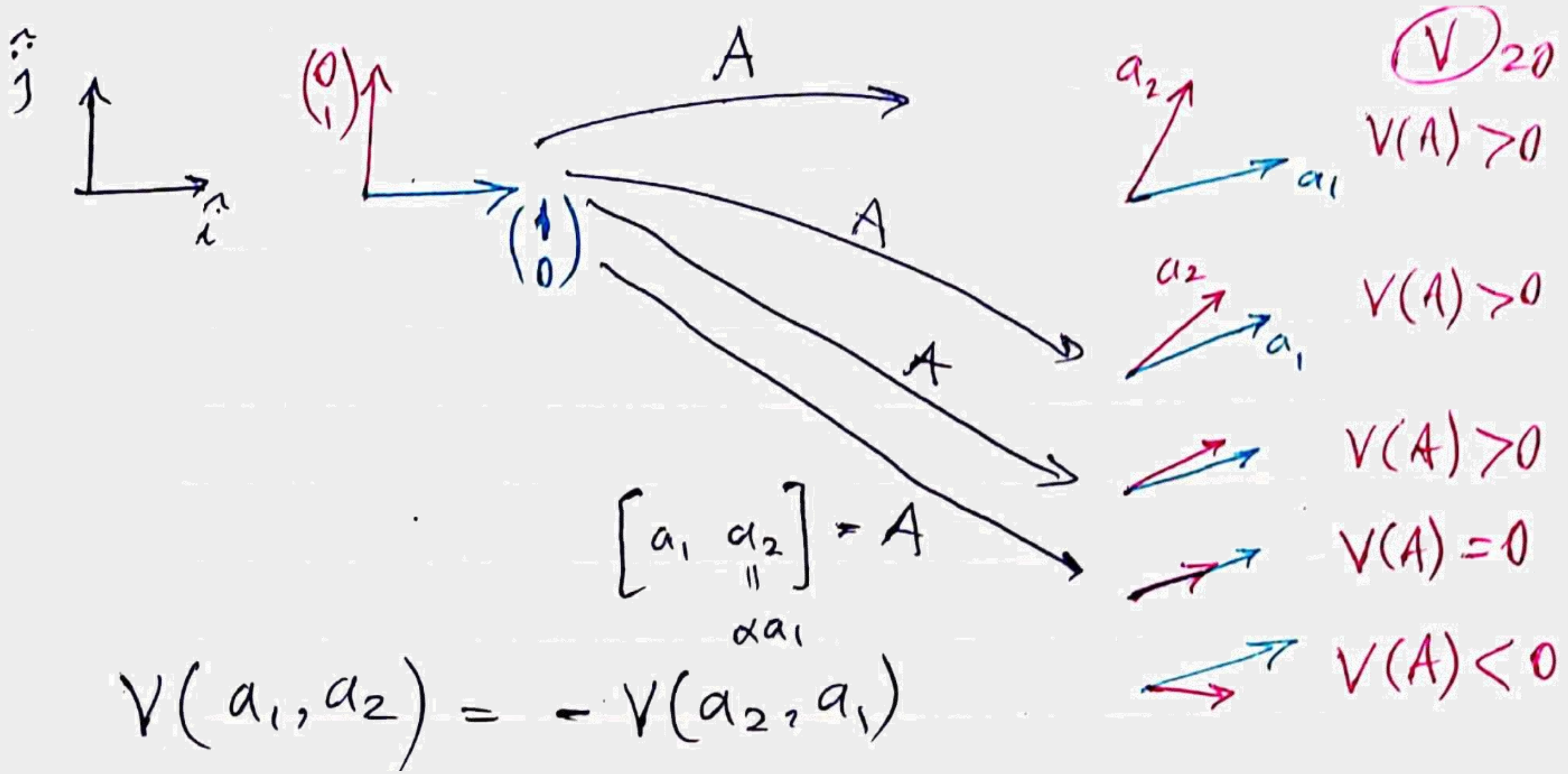
- The volume is a n-alternating multilinear map on all n-parallelepipeds such that the volume of standard unit parallelepiped is one.

$$\frac{\textit{volume of output region}}{\textit{volume of input region}}$$



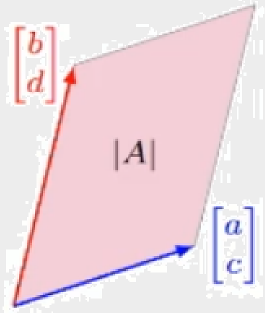
A 2×2 matrix A stretches the unit square (with sides e_1 and e_2) into a parallelogram with sides Ae_1 and Ae_2 (the columns of A). The determinant of A is the area of this parallelogram.

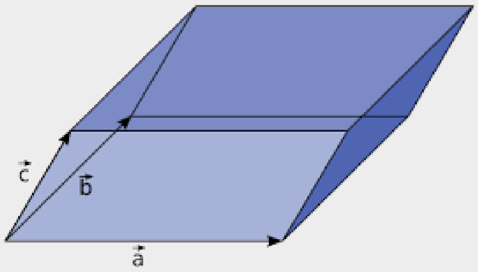
Geometric interpretation



- If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is det(A)
- If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is det(A)
- Examples on board!

Volume of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$??





- Every $n-$dimensional parallelepiped with $\{a_1, \ldots, a_n\}$ as legs is associated with a real number, called its volume which has the following properties:

  1. If we stretch a parallelepiped by multiplying one of its legs by a scalar λ, its volume gets multiplied by λ.
  2. If we add a vector w to i-th legs of a $n-$dimensional parallelepiped with $\{a_1, \ldots, a_i, a_{i+1}, \ldots, a_n\}$, then its volume is the sum of the volume from $\{a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n\}$ and the volume of $\{a_1, \ldots, a_{i-1}, w, a_{i+1}, \ldots, a_n\}$.
  3. The volume changes sign when two legs are exchanged.
  4. The volume of the parallelepiped with $\{e_1, \ldots, e_n\}$ is one.

$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

Bilinear Form: Review and Continue

Bilinear Form over a complex vector space

Definition

Suppose V and W are vector spaces over the same field $\mathbb{C}$. Then a function $\alpha: V \times W \rightarrow \mathbb{C}$ is called a **bilinear form** if it satisfies the following properties:

a) It is linear in its first argument:

i. $\alpha(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = \alpha(\mathbf{v_1}, \mathbf{w}) + \alpha(\mathbf{v_2}, \mathbf{w})$ and

ii. $\alpha(\lambda \mathbf{v_1}, \mathbf{w}) = \lambda\alpha(\mathbf{v_1}, \mathbf{w})$ for all $\lambda \in \mathbb{C}, \mathbf{v_1}, \mathbf{v_2} \in V$, and $\mathbf{w} \in W$.

b) It is conjugate linear in its second argument:

i. $\alpha(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = \alpha(\mathbf{v}, \mathbf{w_1}) + \alpha(\mathbf{v}, \mathbf{w_2})$ and

ii. $\alpha(\mathbf{v}, \lambda \mathbf{w_1}) = \bar{\lambda}\alpha(\mathbf{v}, \mathbf{w_1})$ for all $\lambda \in \mathbb{C}, \mathbf{v} \in V$, and $\mathbf{w_1}, \mathbf{w_2} \in W$.

The set of bilinear forms on $\mathbf{v}$ is denoted by $\mathbf{v}^2$.

Definition

A bilinear form $\alpha \in V^{(2)}$ is called *alternating* if

$$\alpha(v, v) = 0$$

for all $v \in V$. The set of alternating bilinear forms on V is denoted by $V^{(2)}_{\text{alt}}$.

Example

Suppose $\varphi, \tau \in V'$. Then the bilinear form α on V defined by

$$\alpha(u, w) = \varphi(u)\tau(w) - \varphi(w)\tau(u)$$

Alternating bilinear form

Theorem

A bilinear form α on V is alternating if and only if

$$\alpha(u, w) = -\alpha(w, u)$$

for all $u, w \in V$.

Proof

Alternating bilinear form

Theorem

The sets $V_{\text{sym}}^{(2)}$ and $V_{\text{alt}}^{(2)}$ are subspaces of $V^{(2)}$. Furthermore,

$$V^{(2)} = V_{\text{sym}}^{(2)} \oplus V_{\text{alt}}^{(2)}.$$

Proof

Multilinear Form

Definition

Suppose $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_p$ are vector spaces over the same field $\mathbb{F}$. A function

$$f : \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_p \to \mathbb{F}$$

is called a **multilinear form** if, for each $1 \leq j \leq p$ and each $\mathrm{v}_1 \in \mathcal{V}_1, \mathrm{v}_2 \in \mathcal{V}_2, \ldots, \mathrm{v}_p \in \mathcal{V}_p$, it is the case that the function $g : \mathcal{V}_j \to \mathbb{F}$ defined by

$$g(\mathrm{v}) = f\left(\mathrm{v}_1, \ldots, \mathrm{v}_{j-1}, \mathrm{v}, \mathrm{v}_{j+1}, \ldots, \mathrm{v}_p\right) \qquad \textit{for all} \;\; \mathrm{v} \in \mathcal{V}_j$$

is a linear form.

Example

Suppose $\alpha, \rho \in V^{(2)}$. Define a function $\beta : V^4 \to \mathbf{F}$ by

$$\beta(v_1, v_2, v_3, v_4) = \alpha(v_1, v_2)\,\rho(v_3, v_4).$$

Definition

Suppose m is a positive integer.

- An m-linear form α on V is called *alternating* if $\alpha(v_1, \ldots, v_m) = 0$ whenever $v_1, \ldots, v_m$ is a list of vectors in V with $v_j = v_k$ for some two distinct values of j and k in $\{1, \ldots, m\}$.
- $V_{\text{alt}}^{(m)} = \{\alpha \in V^{(m)} : \alpha \text{ is an alternating } m\text{-linear form on } V\}$.

$V_{\text{alt}}^{(m)}$ is a subspace of $V^{(m)}$.

Theorem

An indexed set $S = \{v_1, \dots, v_n\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others.

In fact, if S is linearly dependent and $v_1 \neq 0$, then **some** v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \dots, v_{j-1}$.

Proof

- Does not say that every vector
- Does not say that every vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Alternating multilinear forms and linear dependence

Theorem

Suppose m is a positive integer and α is an alternating m-linear form on V. If $v_1, \ldots, v_m$ is a linearly dependent list in V, then

$$\alpha(v_1, \ldots, v_m) = 0.$$

Proof

no nonzero alternating m–linear forms for $m > \dim V$

Theorem

Suppose $m > \dim V$. Then 0 is the only alternating m-linear form on V.

Proof

Swapping input vectors in an alternating multilinear form

Theorem

Suppose m is a positive integer, α is an alternating m-linear form on V, and $v_1, \dots, v_m$ is a list of vectors in V. Then swapping the vectors in any two slots of $\alpha(v_1, \dots, v_m)$ changes the value of α by a factor of -1.

Proof

Definition

Suppose m is a positive integer.

- A *permutation* of $(1, \ldots, m)$ is a list $(j_1, \ldots, j_m)$ that contains each of the numbers $1, \ldots, m$ exactly once.
- The set of all permutations of $(1, \ldots, m)$ is denoted by $\operatorname{perm} m$.

Definition

The *sign* of a permutation $(j_1, \ldots, j_m)$ is defined by

$$\operatorname{sign}(j_1, \ldots, j_m) = (-1)^N,$$

where N is the number of pairs of integers (k, ℓ) with $1 \leq k < \ell \leq m$ such that k appears after ℓ in the list $(j_1, \ldots, j_m)$.

Example

- The permutation $(1, \ldots, m)$ [no changes in the natural order] has sign 1.
- The only pair of integers (k, ℓ) with $k < \ell$ such that k appears after ℓ in the list $(2, 1, 3, 4)$ is $(1, 2)$. Thus the permutation $(2, 1, 3, 4)$ has sign -1.
- In the permutation $(2, 3, \ldots, m, 1)$, the only pairs (k, ℓ) with $k < \ell$ that appear with changed order are $(1, 2), (1, 3), \ldots, (1, m)$. Because we have $m - 1$ such pairs, the sign of this permutation equals $(-1)^{m-1}$.

Swapping two entries in a permutation

Theorem

Swapping two entries in a permutation multiplies the sign of the permutation by -1.

Proof

Theorem

Suppose m is a positive integer and $\alpha \in V_{\text{alt}}^{(m)}$. Then

$$\alpha(v_{j_1}, \ldots, v_{j_m}) = (\operatorname{sign}(j_1, \ldots, j_m))\alpha(v_1, \ldots, v_m)$$

for every list $v_1, \ldots, v_m$ of vectors in V and all $(j_1, \ldots, j_m) \in \operatorname{perm} m$.

Proof

Formula for $(\dim V)$–linear alternating forms on V

Theorem

Let $n = \dim V$. Suppose $e_1, \ldots, e_n$ is a basis of V and $v_1, \ldots, v_n \in V$. For each $k \in \{1, \ldots, n\}$, let $b_{1,k}, \ldots, b_{n,k} \in \mathbf{F}$ be such that

$$v_k = \sum_{j=1}^{n} b_{j,k} e_j.$$

$$v_1 = \begin{bmatrix} a \\ b \end{bmatrix}, v_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$

Then

$$\alpha(v_1, \ldots, v_n) = \alpha(e_1, \ldots, e_n) \sum_{(j_1, \ldots, j_n) \in \operatorname{perm} n} (\operatorname{sign}(j_1, \ldots, j_n)) b_{j_1,1} \cdots b_{j_n,n}$$

for every alternating n-linear form α on V.

Proof

Homework

Theorem

The vector space $V_{\mathrm{alt}}^{(\dim V)}$ has dimension one.

Proof

Theorem

$$\alpha(v_1, \dots, v_n) = \sum_{(j_1,\dots,j_n)\in \operatorname{perm} n} (\operatorname{sign}(j_1, \dots, j_n))\varphi_{j_1}(v_1)\cdots\varphi_{j_n}(v_n)$$

The verification that α is an n-linear form on V is straightforward.

$$\alpha(e_1, \dots, e_n) = 1$$

Matrix Determinant

Determinant

Definition

Suppose that m is a positive integer and $T \in \mathcal{L}(V)$. For $\alpha \in V_{\text{alt}}^{(m)}$, define $\alpha_T \in V_{\text{alt}}^{(m)}$ by

$$\alpha_T(v_1, \dots, v_m) = \alpha(Tv_1, \dots, Tv_m)$$

for each list $v_1, \dots, v_m$ of vectors in V.

$$\alpha_T = (\det T)\,\alpha$$

Determinant is an alternating multilinear form

Theorem

Suppose that n is a positive integer. The map that takes a list $v_1, \ldots, v_n$ of vectors in $\mathbf{F}^n$ to $\det\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is an alternating n-linear form on $\mathbf{F}^n$.

Proof Let $e_1, \ldots, e_n$ be the standard basis of $\mathbf{F}^n$ and suppose $v_1, \ldots, v_n$ is a list of vectors in $\mathbf{F}^n$. Let $T \in \mathcal{L}(\mathbf{F}^n)$ be the operator such that $Te_k = v_k$ for $k = 1, \ldots, n$. Thus T is the operator whose matrix with respect to $e_1, \ldots, e_n$ is $\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$. Hence $\det\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} = \det T$, by definition of the determinant of a matrix.

Let α be an alternating n-linear form on $\mathbf{F}^n$ such that $\alpha(e_1, \ldots, e_n) = 1$. Then

$$\begin{aligned}
\det\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} &= \det T \\
&= (\det T)\,\alpha(e_1, \ldots, e_n) \\
&= \alpha(Te_1, \ldots, Te_n) \\
&= \alpha(v_1, \ldots, v_n),
\end{aligned}$$

where the third line follows from the definition of the determinant of an operator. The equation above shows that the map that takes a list of vectors $v_1, \ldots, v_n$ in $\mathbf{F}^n$ to $\det\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}$ is the alternating n-linear form α on $\mathbf{F}^n$. ■

Matrix Determinant

Theorem

Suppose that n is a positive integer and A is an n-by-n square matrix. Then

$$\det A = \sum_{(j_1,\dots,j_n)\in \operatorname{perm} n} (\operatorname{sign}(j_1,\dots,j_n)) A_{j_1,1} \cdots A_{j_n,n}.$$

Proof

Definition of Submatrix A_{ij}

Definition

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A

For instance, if

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

A_{12} is

$$A_{12} = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 0 & 7 \\ 0 & -2 & 0 \end{bmatrix}$$

Recursive Definition of Determinant

Definition

The determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j}\det(A_{1j})$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{1+n}a_{1n}\det(A_{1n})$$

$$= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det(A_{1j})$$

Recursive Definition of Determinant

- 2 × 2 matrix $\qquad |A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \left| A_{ij} \right| \qquad i = 1$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow |A| = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}|$$

$$= a \begin{vmatrix} \square & \square \\ \square & d \end{vmatrix} - b \begin{vmatrix} \square & \square \\ c & \square \end{vmatrix}$$

$$= ad - bc$$

Example

$$\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = (-1) \times (1) - (2) \times (-3) = 5$$

Recursive Definition of Determinant

- ❑ 3×3 matrix $\qquad |A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \qquad i = 1$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \quad |A| = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}| + (-1)^{1+3} a_{13} |A_{13}|$$

$$= a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + cdh - afh - bdi - ceg$$

Recursive Definition of Determinant

Example

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{vmatrix} = -5 + 0 + 6 - (25 + 12 + 0) = -36$$

Cofactor

Definition

Given $A = [a_{ij}]$, the (i, j)-cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

Then

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Which is a cofactor expansion across the first row of A.

Cofactor Expansion

Important

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactor is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Example

$$A = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} 5 & 4 \\ 3 & -1 \end{vmatrix} - 0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 5 \\ 5 & 3 \end{vmatrix} = -36$$

$$|A| = -0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 5 \times \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = -36$$

Determinant Properties

Properties

- (1) If one row or column is zero, then determinant is zero

$$\begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

- Determinant of zero matrix is⋯

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

- (2) If two rows or columns of matrix are same, then determinant is zero.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} - (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

$$|A| = -1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} + (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

Properties

- (3) If two rows or columns of matrix are interchanged, the sign of determinant is changes!
- $(4)\ \det(I) = 1$

Properties

- (5) Row and Column Operations
  - If a multiple of one row/column of A is added to another row/column to produce a matrix B, then $\det(A) = \det(B)$.

Proof?

Example

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 1 & 3 \\ 1 & -1 & 4 \end{vmatrix}$$

- (6) If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \qquad \begin{vmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{vmatrix} = abc$$

  - Determinant of identity matrix is…

- U is unitary, so that $|\det(U)|=I$

- (7) If a column or row is multiply to k then determinant is multiply to k.

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$$

$$\begin{vmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = ka_{11}C_{11} + \cdots + ka_{1n}C_{1n} = k\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

- $|kA_{n\times n}| = k^n|A_{n\times n}|$

Properties

- (8) If a row/column is multiple of another row/column then determinant is ⋯..

Properties

- (9) If columns/rows of matrix are linear dependent then its determinant is zero

- (10) If columns/rows of matrix are linear dependent if and only if its determinant is zero.

Theorem

A square matrix A is invertible if and only if $\det(A) \neq 0$

Example

Compute $\det(A)$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

Note

Row operations

Let A be a square matrix.

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$

b. If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$

c. If one row of A is multiplied by k to produce B, then $\det(B) = k.\det(A)$

Echelon form

Example

Compute $\det(A)$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

Determinant of Transpose

Theorem

if A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$

Multiplicative Property

Theorem

if A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$

Important

In general, $\det(A + B) \neq \det(A) + \det(B)$

- The determinant of the inverse of an invertible matrix is the inverse of the determinant

$$AA^{-1} = I \Rightarrow |AA^{-1}| = |I| = 1 \Rightarrow |A||A^{-1}| = 1 \Rightarrow |A^{-1}| = |A|^{-1}$$

- The determinant of orthogonal matrix is …

Cramer's Rule

- $Ax = B$ and A is invertible $\qquad A = [a_1 \quad \dots \quad a_n] \qquad I = [e_1 \quad \dots \quad e_n]$

$$AI = A \implies A[e_1 \quad \dots \quad e_n] = [Ae_1 \quad \dots \quad Ae_n] = [a_1 \quad \dots \quad a_n]$$

$$A\,\overbrace{[e_1 \quad e_2 \quad \dots \quad x \quad \cdots \quad e_n]}^{I_j(x)} = [Ae_1 \quad Ae_2 \quad \dots \quad Ax \quad \cdots \quad Ae_n]$$

$$= \underbrace{[a_1 \quad a_2 \quad \dots \quad b \quad \cdots \quad a_n]}_{A_j(b)}$$

$$|I_2(x)| = \begin{vmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{vmatrix} = x_2 \implies |I_j(x)| = x_j$$

$$AI_j(x) = A_j(b) \implies |A||I_j(x)| = |A_j(b)| \implies x_j = \frac{|A_j(b)|}{|A|}$$

Cramer's Rule

Note

Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{|A_i(\mathbf{b})|}{|A|}, \qquad i = 1,2,\dots,n$$

Example

$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 2x_1 - 3x_2 + x_3 = -1 \end{cases} \implies x_2 = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 2 & -3 & 1 \end{vmatrix}} = \frac{-12}{-3} = 4$$

A Formula for A^{-1}

The j–th column of A^{-1} is a vector x that satisfies $\quad Ax = e_j$

By Cramer's rule $\quad \{(\mathrm{i},\mathrm{j}) - \text{entry of } A^{-1}\} = x_i = \dfrac{|A_i(e_j)|}{|A|}$

$$|A_i(e_j)| = (-1)^{i+j}|A_{ji}|$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

The matrix of cofactors is called the adjugate (or classical adjoint) of A, denoted by adj A.

$$\begin{cases} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ [C_{ij}] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \end{cases} \Rightarrow A^{-1} = \frac{1}{|A|}[C_{ij}]^T = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

A Formula for A^{-1}

Important

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{|A|}\, adj\ A$$

Example

Show that the determinant, $det\colon \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$ is not a linear transformation when $n \geq 2$

Note

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{area\ of\ T(S)\} = |\det A| . \{area\ of\ S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{volume\ of\ T(S)\} = |\det A| . \{volume\ of\ S\}$$

Reference

- Chapter 3: Linear Algebra and Its Applications, David C. Lay.
- Chapter 9: Part B and C: Linear Algebra Done Right, Sheldon Axler.